Casimir densities for a spherical shell in the global monopole background

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Abstract

We investigate the vacuum expectation values for the energy-momentum tensor of a massive scalar field with general curvature coupling and obeying the Robin boundary condition on a spherical shell in the $D+1$-dimensional global monopole background. The expressions are derived for the Wightman function, the vacuum expectation values of the field square, the vacuum energy density, radial and azimuthal stress components in both regions inside and outside the shell. A regularization procedure is carried out by making use of the generalized Abel-Plana formula for the series over zeros of cylinder functions. This formula allows us to extract from the vacuum expectation values the parts due to the global monopole gravitational field in the situation without a boundary, and to present the boundary induced parts in terms of exponentially convergent integrals, useful, in particular, for numerical calculations. The asymptotic behavior of the vacuum densities is investigated near the sphere surface and at large distances. We show that for small values of the parameter describing the solid angle deficit in global monopole geometry the boundary induced vacuum stresses are strongly anisotropic.

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1 Introduction

The Casimir effect is one of the most interesting manifestations of the nontrivial properties of the vacuum state in quantum field theory. Since its first prediction by Casimir in 1948 [1] this effect has been investigated for various cases of boundary geometries and different fields (see [2]–[7] and references therein). Historically, the investigation of the Casimir effect for a perfectly conducting spherical shell was motivated by the Casimir semiclassical model of an electron. It has been shown by Boyer [8] that the Casimir energy for the sphere is positive, implying a repulsive force. This result was later reconsidered by a number of authors [9]–[12]. More recently new methods have been developed for this problem including direct mode summation techniques based on the zeta function regularization scheme [13]–[27]. In Refs. [28, 29] the Casimir effect is considered for a massless scalar field with the Dirichlet boundary condition on a spherical shell with false/true vacuum inside/outside the shell. Recently, the Casimir effect for spherical boundaries on de Sitter and anti de Sitter bulks is also investigated [30]–[32]. In the investigations of the Casimir effect the calculation of the local densities of the vacuum characteristics is of special interest. In particular, these include the vacuum expectation values of the energy-momentum tensor. In addition to describing the physical structure of the quantum field at a given point, the energy-momentum tensor acts as the source of gravity in the Einstein equations. It therefore plays an important role in modeling a self-consistent dynamics involving the gravitational field [33]. Investigation of the energy distribution inside a perfectly reflecting spherical shell was made in [34] in the case of QED and in [35] for QCD. The distributions of the other components for the energy-momentum tensor of the electromagnetic field inside as well as outside the shell, and in the region between two concentric spherical shells are investigated in Refs. [36]–[39]. Recently, in Ref. [40] the vacuum expectation values of the energy-momentum tensor are evaluated for a massive scalar field with general curvature coupling parameter, satisfying the Robin boundary condition on spherically symmetric boundaries in D-dimensional space.

The Casimir effect can be viewed as a polarization of vacuum by boundary conditions. Another type of vacuum polarization arises in the case of external gravitational field. In this paper we study a situation when both types of sources for the polarization are present. Namely, we consider a global monopole spacetime with a concentric spherical shell. Topological defects have attracted a great deal of attention because of their relevance to a number of different areas ranging from condensed matter to structure formation (for a review see [41]). In the context of hot big bang cosmology, the unified theories of the fundamental interactions predict that the universe passes through a sequence of phase transitions. These phase transitions might have given rise to several kinds of topological defects depending on the nature of the symmetry that is broken [42]. If a global SO(3) symmetry of a triplet scalar field is broken, the point like defects called global monopoles are believed to be formed. The simplified global monopole was introduced by Sokolov and Starobinsky [43]. The gravitational effects of the global monopole were studied in Ref. [44], where a solution is presented which describes a global monopole at large radial distances. The quantum vacuum effects of the matter fields on the global monopole background have been considered in Refs. [45]–[51]. The effects produced by the non-zero temperature are investigated as well [49]. The zeta function and the Casimir energy for a spherical boundary in this background are calculated in [50, 51].

In this paper the positive frequency Wightman function, the vacuum expectation values of the field square and energy-momentum tensor are investigated for a massive scalar field with general curvature coupling parameter ξ, satisfying the Robin boundary condition on a spherical shell in the $D+1$-dimensional spacetime of a point-like global monopole. As special cases they include the results for the Diriclet, Neumann, TM, and conformally invariant Hawking boundary conditions. To evaluate the corresponding bilinear field products we use the mode sum method in combination with the summation formulae from Ref. [52] (see also [53]). These formulae allow

(i) to extract from vacuum expectation values the parts due to the global monopole background without boundaries, and (ii) to present the boundary induced parts in terms of exponentially convergent integrals involving the modified Bessel functions. We have organized the paper as follows. In the next section we consider the vacuum inside a sphere and derive formulae for the Wightman function, the expectation values of the field square, energy density and stresses on the global monopole background. The behavior of these quantities is investigated near the boundary, at the sphere centre and for small values of the parameter describing the solid angle deficit in the global monopole geometry. Section 3 is devoted to the vacuum expectation values for the region outside a sphere. Section 4 concludes the main results of the paper.

2 Wightman function, vacuum expectation values of the field square and energy-momentum tensor inside a spherical shell

2.1 Wightman function

Consider a real scalar field φ with curvature coupling parameter ξ on a $D+1$-dimensional global monopole background. In the hyperspherical polar coordinates $(r, \vartheta, \phi) \equiv (r, \theta_1, \theta_2, \dots \theta_n, \phi)$, $n = D-2$, the corresponding line element has the form

$$ds^2 = dt^2 - dr^2 - \sigma^2 r^2 d\Omega_D^2, \tag{1}$$

where $d\Omega_D^2$ is the line element on the surface of the unit sphere in D-dimensional Euclidean space, the parameter σ is smaller than unity and is related to the symmetry breaking energy scale in the theory. The solid angle corresponding to Eq. (1) is $\sigma^2 S_D$ with $S_D = 2\pi^{D/2}/\Gamma(D/2)$ being the total area of the surface of the unit sphere in D-dimensional Euclidean space. This leads to the solid angle deficit $(1-\sigma^2)S_D$ in the spacetime given by line element (1). The field equation has the form

$$\left(\nabla_i \nabla^i + m^2 + \xi R\right) \varphi = 0, \tag{2}$$

where R is the scalar curvature for the background space-time, m is the mass for the field quanta, ∇_i is the covariant derivative operator associated with the metric corresponding to line element (1). The values of the curvature coupling parameter $\xi = 0$, and $\xi = \xi_D$ with $\xi_D \equiv (D-1)/4D$ correspond to the minimal and conformal couplings, respectively. It is convenient for later use to write the corresponding metric energy-momentum tensor in the form

$$T_{ik} = \nabla_i \varphi \nabla_k \varphi + \left[\left(\xi - \frac{1}{4}\right) g_{ik} \nabla_l \nabla^l - \xi \nabla_i \nabla_k - \xi R_{ik}\right] \varphi^2. \tag{3}$$

The nonzero components of the Ricci tensor and Ricci scalar for the metric corresponding to line element (1) are given by expressions

$$R_2^2 = R_3^3 = \cdots = R_D^D = n\frac{\sigma^2 - 1}{\sigma^2 r^2}, \quad R = n(n+1)\frac{\sigma^2 - 1}{\sigma^2 r^2}, \tag{4}$$

where the indices $2, 3, \dots, D$ correspond to the coordinates $\theta_1, \theta_2, \dots, \phi$ respectively, and we adopt the convention of Birrell and Davies [33] for the curvature tensor. Note that for $\sigma \neq 1$ the geometry is singular at the origin (point-like monopole), $r = 0$.

In this paper we are interested in the vacuum expectation values (VEVs) of the energy-momentum tensor (3) on background of the geometry described by (1), assuming that the field satisfies the Robin boundary condition

$$\left(A_1 + B_1 n^i \nabla_i\right) \varphi(x) = 0 \tag{5}$$

on the sphere of radius a, concentric with monopole. Here A_1 and B_1 are constants, and $n^i = (0, n^1, 0, 0)$ is the unit normal to the sphere, $n^1 = -1, 1$ for the interior and exterior regions respectively. The imposition of this boundary condition on the quantum field $\varphi(x)$ leads to the modification of the spectrum for the zero-point fluctuations and results in the shift in VEVs for physical quantities. In particular, vacuum forces arise acting on constraining boundary. This is the familiar Casimir effect. By virtue of Eq. (3) for the VEV of the energy-momentum tensor we have

$$\langle 0|T_{ik}(x)|0\rangle = \lim_{x'\to x} \partial_i\partial'_k \langle 0|\varphi(x)\varphi(x')|0\rangle + \left[\left(\xi - \frac{1}{4}\right) g_{ik}\nabla_l\nabla^l - \xi\nabla_i\nabla_k - \xi R_{ik}\right] \langle 0|\varphi^2(x)|0\rangle, \quad (6)$$

where $|0\rangle$ is the amplitude for the corresponding vacuum state. Note that the expectation value $\langle 0|\varphi(x)\varphi(x')|0\rangle \equiv G^+(x, x')$ is known as a positive frequency Wightman function. To derive the expression for the regularized VEV of the field bilinear product we will use the mode summation method. By expanding the field operator over eigenfunctions and using the commutation rules one can see that

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \sum_\alpha \varphi_\alpha(x)\varphi^*_\alpha(x'), \quad (7)$$

where $\{\varphi_\alpha(x), \varphi^*_\alpha(x')\}$ is a complete orthonormal set of positive and negative frequency solutions to the field equation with quantum numbers α, satisfying boundary condition (5).

In the hyperspherical coordinates, for the region inside the sphere the complete set of solutions to Eq. (2) with scalar curvature from (4), has the form

$$\varphi_\alpha(x) = \beta_\alpha r^{-n/2} J_{\nu_l}(\lambda r) Y(m_k; \vartheta, \phi) e^{-i\omega t}, \ l = 0, 1, 2, \dots, \quad (8)$$

where $m_k = (m_0 \equiv l, m_1, \dots, m_n)$, and $m_1, m_2, \dots, m_n$ are integers such that

$$0 \le m_{n-1} \le m_{n-2} \le \cdots \le m_1 \le l, \quad -m_{n-1} \le m_n \le m_{n-1}, \quad (9)$$

$J_\nu(z)$ is the Bessel function, and $Y(m_k; \vartheta, \phi)$ is the surface harmonic of degree l (see [54], Section 11.2). In Eq. (8) we use the following notation

$$\begin{aligned} \nu_l &= \frac{1}{\sigma}\left[\left(l + \frac{n}{2}\right)^2 + (1 - \sigma^2)n(n+1)(\xi - \xi_{D-1})\right]^{1/2}, \\ \lambda &= \sqrt{\omega^2 - m^2}. \end{aligned} \quad (10)$$

In the following consideration we will assume that ν_l^2 is non-negative. This corresponds to the restriction on the values of the curvature coupling parameter for $n > 0$, given by the condition

$$\xi \ge -\frac{n}{4(n+1)(\sigma^{-2} - 1)}. \quad (11)$$

Note that this condition is satisfied by the most important special cases of the minimal and conformal couplings. The coefficients β_α in Eq. (8) can be found from the normalization condition

$$\int |\varphi_\alpha(x)|^2 \sqrt{-g} dV = \frac{1}{2\omega}, \quad (12)$$

where the integration goes over the region inside the sphere. Substituting eigenfunctions (8), and using the relation

$$\int |Y(m_k; \vartheta, \phi)|^2 d\Omega = N(m_k) \quad (13)$$

(the explicit form for $N(m_k)$ is given in [54], Section 11.3, and will not be necessary for the following considerations in this paper) for the spherical harmonics and the value for the standard integral involving the square of the Bessel function, one finds

$$\beta_\alpha^2 = \frac{\lambda T_{\nu_l}(\lambda a)}{N(m_k)\omega a \sigma^{D-1}}, \tag{14}$$

with the notation

$$T_\nu(z) = \frac{z}{(z^2-\nu^2)J_\nu^2(z) + z^2 J_\nu'^2(z)}. \tag{15}$$

From boundary condition (5) on the sphere surface for eigenfunctions (8) one sees that the possible values for the frequency have to be solutions to the following equation

$$AJ_{\nu_l}(z) + BzJ_{\nu_l}'(z) = 0, \quad z = \lambda a, \quad A = A_1 - \frac{n}{2}B, \quad B = \frac{n^1}{a}B_1, \tag{16}$$

where $n^1 = -1$ for the region inside the sphere. It is well known (see, e.g., [54, 55]) that for real A, B, and $\nu_l > -1$ all roots of this equation are simple and real, except the case $A/B < -\nu_l$ when there are two purely imaginary zeros. In the following we will assume values of A/B for which all roots are real, $A/B \geq -\nu_l$. In terms of the coefficients in (5), it is sufficient to require

$$\frac{A_1 a}{B_1} \leq \nu_0 - \frac{n}{2}, \quad \nu_0 = \frac{n}{2}\left[1 + 4(\sigma^{-2}-1)\frac{n+1}{n}\xi\right]^{1/2}. \tag{17}$$

Note that this condition is satisfied for the most important special cases of Dirichlet scalar, Neumann scalar ($A/B = 1 - D/2$) with the curvature coupling parameter $\xi \geq 0$, and for a scalar field with the TM type boundary condition ($A/B = D/2 - 1$).

Let $\lambda_{\nu_l,k}$, $k = 1, 2, \ldots$, be the positive zeros of the function $AJ_{\nu_l}(z) + BzJ_{\nu_l}'(z)$, arranged in ascending order. The corresponding eigenfrequencies $\omega = \omega_{\nu_l,k}$ are related to these zeros as $\omega_{\nu_l,k} = \sqrt{\lambda_{\nu_l,k}^2/a^2 + m^2}$. Substituting Eq. (8) into Eq. (7) and using the addition formula for the spherical harmonics (see [54], Section 11.4), one obtains

$$\begin{aligned}\langle 0|\varphi(x)\varphi(x')|0\rangle &= \frac{(rr')^{-n/2}}{naS_D\sigma^{D-1}}\sum_{l=0}^{\infty}(2l+n)C_l^{n/2}(\cos\theta) \\ &\times \sum_{k=1}^{\infty}\frac{\lambda_{\nu_l,k}T_{\nu_l}(\lambda_{\nu_l,k})}{\sqrt{\lambda_{\nu_l,k}^2 + m^2a^2}}J_{\nu_l}(\lambda_{\nu_l,k}r/a)J_{\nu_l}(\lambda_{\nu_l,k}r'/a)e^{i\omega_{\nu_l,k}(t'-t)},\end{aligned} \tag{18}$$

where $C_p^q(x)$ is the Gegenbauer or ultraspherical polynomial of degree p and order q, and θ is the angle between directions (ϑ, ϕ) and (ϑ', ϕ'). To sum over k we will use the generalized Abel-Plana summation formula [52, 53]

$$\begin{aligned}2\sum_{k=1}^{\infty}T_\nu(\lambda_{\nu,k})f(\lambda_{\nu,k}) &= \int_0^\infty f(x)dx + \frac{\pi}{2}\mathrm{Res}_{z=0}f(z)\frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)} \\ &- \frac{1}{\pi}\int_0^\infty dx\frac{\bar{K}_\nu(x)}{\bar{I}_\nu(x)}\left[e^{-\nu\pi i}f(xe^{\pi i/2}) + e^{\nu\pi i}f(xe^{-\pi i/2})\right],\end{aligned} \tag{19}$$

where $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions, and for a given function $F(z)$ we use the notation

$$\bar{F}(z) \equiv AF(z) + BzF'(z). \tag{20}$$

By taking in this formula $\nu = 1/2$, $A = 1$, $B = 0$, as a particular case we receive the Abel-Plana formula [56]. Formula (19) was applied previously to a number of Casimir problems for spherically [38, 39, 40] and cylindrically [57, 58] symmetric boundaries on the Minkowski background, and for a wedge with and without circular outer boundary [59]. Note that formula (19) can be generalized in the case of the existence of purely imaginary zeros for the function $\bar{J}_\nu(z)$ by adding the corresponding residue term and taking the principal value of the integral on the right (see Ref. [53]). As it has been mentioned earlier, in this paper we will assume values of A/B for which all solutions to Eq. (16) are real. Applying to the sum over k in Eq. (18) formula (19), the Wightman function can be presented in the form

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \langle 0_{\rm m}|\varphi(x)\varphi(x')|0_{\rm m}\rangle + \langle\varphi(x)\varphi(x')\rangle_b, \tag{21}$$

where

$$\langle 0_{\rm m}|\varphi(x)\varphi(x')|0_{\rm m}\rangle = \frac{1}{2nS_D\sigma^{D-1}}\sum_{l=0}^{\infty}\frac{2l+n}{(rr')^{n/2}}C_l^{n/2}(\cos\theta)\int_0^\infty dz\,\frac{ze^{i\sqrt{z^2+m^2}(t'-t)}}{\sqrt{z^2+m^2}}J_{\nu_l}(zr)J_{\nu_l}(zr'), \tag{22}$$

and

$$\begin{aligned}\langle\varphi(x)\varphi(x')\rangle_b &= -\frac{1}{\pi naS_D\sigma^{D-1}}\sum_{l=0}^{\infty}\frac{2l+n}{(rr')^{n/2}}C_l^{n/2}(\cos\theta)\times \\ &\quad \int_{ma}^\infty dz\,z\frac{\bar{K}_{\nu_l}(z)}{\bar{I}_{\nu_l}(z)}\frac{I_{\nu_l}(zr/a)I_{\nu_l}(zr'/a)}{\sqrt{z^2-m^2a^2}}\cosh\left[\sqrt{z^2/a^2-m^2}(t'-t)\right].\end{aligned} \tag{23}$$

To obtain Eq. (23) we have used the result that the difference of the radicals is nonzero above the branch point only. The conditions for the formula (19) to be valid in the case of the sum over k in Eq. (18) are satisfied if $r+r'+|t-t'|<2a$. The contribution of the term (22) to the VEV does not depend on a, whereas the contribution of the term (23) tends to zero as $a\to\infty$. It follows from here that expression (22) is the Wightman function for the unbounded global monopole space with $|0_{\rm m}\rangle$ being the amplitude for the corresponding vacuum. This can be seen also by explicit evaluation of the mode sum using the eigenfunctions for the global monopole spacetime without a boundary. Note that for $\sigma=1$, ν_l is equal to $l+1/2$ and the sum over l in Eq. (22) can be summarized using the Gegenbauer addition theorem for the Bessel function [60]. Evaluating the remaining integral involving the Bessel function [61], we obtain the standard expression for the $D+1$-dimensional Minkowskian Wightman function. As we have seen, the application of the generalized Abel-Plana formula allows us to extract from the bilinear field product the contribution due to the unbounded monopole spacetime, and the term (23) can be interpreted as the part of the VEV induced by a spherical boundary. In the massless case by using the formula for the integral involving the Bessel functions, for the Wightman function (22) can be presented as

$$\langle 0_{\rm m}|\varphi(x)\varphi(x')|0_{\rm m}\rangle = \frac{1}{2\pi nS_D}\sum_{l=0}^{\infty}\frac{(2l+n)C_l^{n/2}(\cos\theta)}{(\sigma^2rr')^{(n+1)/2}}Q_{\nu_l-1/2}\left(\frac{r^2+r'^2-(t-t')^2+i\epsilon}{2rr'}\right), \tag{24}$$

where Q_{ν_l} is the Legendre function of second kind, $\epsilon>0$ for $t>t'$, and $\epsilon<0$ for $t<t'$. Note that similar expression for the Euclidean scalar Green function is derived in Ref. [46] for $D=3$ and in Ref. [48] for an arbitrary D. The corresponding VEVs of the energy-momentum tensor are investigated in Refs. [45]–[49].

2.2 VEV for the field square

The VEV of the field square is obtained computing the Wightman function in the coincidence limit $x'\to x$. In this limit expression (21) gives a divergent result and some renormalization procedure is needed. As we will see below for $0<r<a$ the divergences are contained in the first summand on the right of Eq. (21) only. Hence, the renormalization procedure for the local characteristics of the vacuum, such as field square and energy-momentum tensor, is the same as for the global monopole geometry without a boundary. This procedure is discussed in a number of papers (see [45]–[51]) and is a useful illustration for the general renormalization prescription on curved backgrounds (see, for instance, [33]). In accordance with this prescription, for the renormalization we must subtract from (22) the corresponding De Witt – Schwinger expansion involving the terms up to order D. For a massless field the renormalized value of the field square has the following general form

$$\langle 0_{\rm m}|\varphi^2(x)|0_{\rm m}\rangle_{\rm ren}=\frac{1}{r^{D-1}}\left[A^{(D)}(\sigma,\xi)+B^{(D)}(\sigma,\xi)\ln(\mu r)\right], \tag{25}$$

where the arbitrary mass scale μ corresponds to the ambiguity in the renormalization procedure. Note that this ambiguity is absent for a spacetime of odd dimension and, hence, $B^{(D)}=0$ for even D. Because the dependence of the order of the Legendre function in Eq. (24) on the parameter σ is not a simple one, it is not possible to obtain closed expression for the coefficients $A^{(D)}$ and $B^{(D)}$. For small values $1-\sigma^2$ approximate expressions are derived in Ref. [48] for $D=4$ and $D=5$. In this paper our main interest are the parts in the VEVs induced by the presence of a spherical shell and below we will concentrate on these quantities.

Using the relation

$$C_l^{n/2}(1)=\frac{\Gamma(l+n)}{\Gamma(n)l!}, \tag{26}$$

for the corresponding boundary part in the VEV of the field square we get

$$\langle\varphi^2(x)\rangle_b=-\frac{1}{\pi a r^n S_D\sigma^{D-1}}\sum_{l=0}^{\infty}D_l\int_{ma}^{\infty}dz\,\frac{\bar{K}_{\nu_l}(z)}{\bar{I}_{\nu_l}(z)}\frac{zI_{\nu_l}^2(zr/a)}{\sqrt{z^2-m^2a^2}}, \tag{27}$$

where

$$D_l=(2l+D-2)\frac{\Gamma(l+D-2)}{\Gamma(D-1)\,l!} \tag{28}$$

is the degeneracy of each angular mode with given l. From the well known properties of the modified Bessel functions it follows that in the case of a Dirichlet scalar this quantity is negative and for $\xi\geq 0$ is a monotonic decreasing function on r. As it has been noted earlier, expression (27) is finite for all values $0<r<a$. For a given l and large z the subintegrand behaves as $e^{2z(r/a-1)}/z$ and, hence, the integral converges when $r<a$. For large values l, introducing a new integration variable $y=z/\nu_l$ in the integral of Eq. (27) and using the uniform asymptotic expansions for the modified Bessel functions [60], it can be seen that to the leading order over l the subintegrand multiplied by D_l behaves as

$$l^n\frac{\exp\left\{2\nu_l\left[\eta(zr/a)-\eta(z)\right]\right\}}{\sqrt{1+(zr/a)^2}},\quad \eta(z)=\sqrt{1+z^2}+\ln\frac{z}{1+\sqrt{1+z^2}}, \tag{29}$$

and, hence, the both integral and sum are convergent for $r<a$ and diverge at $r=a$. For the points near the sphere the leading term of the corresponding asymptotic expansion over $1/(a-r)$ has the form

$$\langle\varphi^2(x)\rangle_b\approx\frac{(1-2\delta_{B0})\Gamma\left(\frac{D-1}{2}\right)}{(4\pi)^{(D+1)/2}(a-r)^{D-1}}. \tag{30}$$

This term does not depend on the mass and parameter σ and is the same as that for a sphere on the Minkowski bulk. As the purely gravitational part (25) is finite for $r=a$, we see that near the sphere surface the VEV of the field square is dominated by the boundary induced part, Eq. (27).

In the limit $r \to 0$ the main contribution into Eq. (27) comes from the $l = 0$ summand with the leading term

$$\langle \varphi^2(x) \rangle_b \approx -\frac{\Gamma^{-2}(\nu_0+1)}{2^{2\nu_0}\pi a^{D-1} S_D \sigma^{D-1}} \left(\frac{r}{a}\right)^{2\nu_0-n} \int_{ma}^{\infty} dz \frac{z^{2\nu_0+1}}{\sqrt{z^2-m^2a^2}} \frac{\bar{K}_{\nu_0}(z)}{\bar{I}_{\nu_0}(z)}, \quad r \to 0, \tag{31}$$

where ν_0 is given by formula (17). As a result the boundary induced VEV (27) is zero at the sphere centre for $(\sigma^{-2}-1)\xi > 0$, non-zero constant for $(\sigma^{-2}-1)\xi = 0$, and is infinite for $(\sigma^{-2}-1)\xi < 0$. As it follows from expressions (25) and (31), for $r \ll a$ the total VEV is dominated by the purely gravitational part.

Now we turn to the limit $\sigma \ll 1$ for a fixed value $r < a$. To satisfy condition (11) we will assume that $\xi \geq 0$. For $\xi > 0$, from Eq. (10) one has $\nu_l \gg 1$, and after introducing in Eq. (27) a new integration variable $y = z/\nu_l$, we can replace the modified Bessel function by their uniform asymptotic expansions for large values of the order. The integral over y can be estimated by using the Laplace method. The main contribution to the sum over l comes from the summand with $l = 0$, and to the leading order we receive

$$\langle \varphi^2(x) \rangle_b \approx \frac{(1-2\delta_{B0})\exp\left[-2\tilde{\nu}\ln(a/r)\right]}{(8\pi\tilde{\nu})^{1/2} r^n S_D \sigma^{D-1}\sqrt{a^2-r^2}}, \quad \tilde{\nu} = \frac{1}{\sigma}\sqrt{n(n+1)\xi}, \quad \sigma \ll 1. \tag{32}$$

For $\xi = 0$ and $\sigma \ll 1$ for the terms with $l \neq 0$ one has $\nu_l \gg 1$. The corresponding contribution can be estimated by the way similar to that in the previous case. This contribution is exponentially small. For the summand with $l = 0$ to the leading order over σ we have $\nu_l = n/2$ and, hence,

$$\langle \varphi^2(x) \rangle_b \approx -\frac{1}{\pi a r^n S_D \sigma^{D-1}} \int_{ma}^{\infty} dz \frac{\bar{K}_{n/2}(z)}{\bar{I}_{n/2}(z)} \frac{z I_{n/2}^2(zr/a)}{\sqrt{z^2-m^2a^2}}, \quad \xi = 0, \quad \sigma \ll 1. \tag{33}$$

Note that for odd values n the modified Bessel functions in this formula are expressed via the elementary functions. In Fig. 1 (left panel) we have plotted the dependence of the boundary induced VEV (27) on r/a for the region inside the sphere in the case of a conformally coupled ($\xi = \xi_D$) massless Dirichlet scalar in $D = 3$ spatial dimensions. The separate curves correspond to the values $\sigma = 1$ (a), $\sigma = 0.5$ (b), $\sigma = 0.2$ (c).

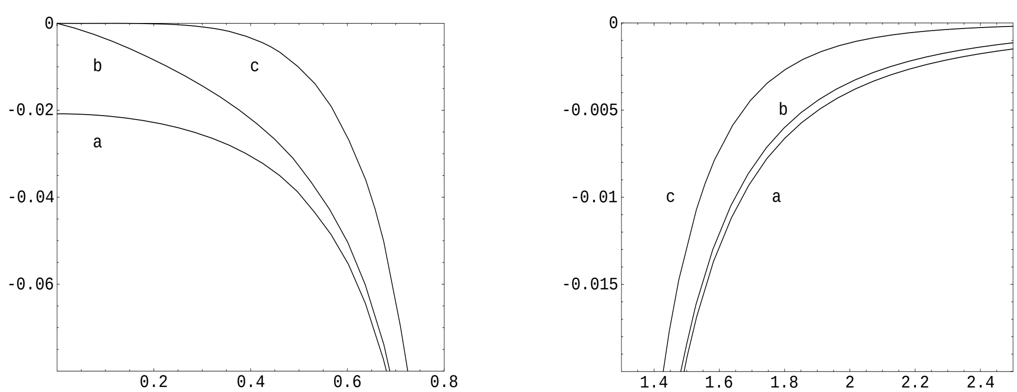

Figure 1: The sphere induced VEV $a^{D-1}\langle \varphi^2(x) \rangle_b$ as a function on r/a in the case of a conformally coupled massless $D = 3$ Dirichlet scalar for the regions inside (left panel) and outside (right panel) the sphere. The curves are plotted for $\sigma = 1$ (a), $\sigma = 0.5$ (b), $\sigma = 0.2$ (c)

2.3 Vacuum energy-momentum tensor inside a sphere

Substituting the Wightman function (21) with expressions (22) and (23) into Eq. (6), for the VEV of the energy-momentum tensor inside a spherical shell one finds

$$\langle 0|T_i^k|0\rangle = \mathrm{diag}\,(\varepsilon, -p, -p_\perp, \ldots, -p_\perp)\,, \tag{34}$$

where the vacuum energy density ε and the effective pressures in radial, p, and azimuthal, $p_\perp$, directions are functions of the radial coordinate only. Similar to the Wightman function, the components of the vacuum energy-momentum tensor can be presented in the form

$$q = q_m + q_b, \quad q = \varepsilon,\, p,\, p_\perp, \tag{35}$$

where q_m are the corresponding quantities for the monopole geometry when the boundary is absent, and quantities q_b are induced by the presence of the spherical shell. As a consequence of the continuity equation $\nabla_k \langle 0|T_i^k|0\rangle = 0$, these functions are related by the equation

$$r\frac{dp_i}{dr} + (D-1)(p_i - p_{\perp i}) = 0, \quad i = m, b. \tag{36}$$

For massless fields the VEV of the energy-momentum tensor for the global monopole geometry without boundaries are investigated in Refs. [45]–[48]. The corresponding renormalized components have the structure similar to that given in Eq. (25) for the field square:

$$q_m = \frac{1}{r^{D+1}}\left[q_m^{(1)} + q_m^{(2)}\ln(\mu r)\right], \tag{37}$$

where the coefficients $q_m^{(1)}$, $q_m^{(2)}$ depend only on the parameters σ and ξ, and $q_m^{(2)} = 0$ in even dimensions D. From equation (36) the following relations between these coefficients are obtained

$$p_m^{(1)} = -\frac{D-1}{2}p_{\perp m}^{(1)} + \frac{1}{2}p_m^{(2)}, \quad p_m^{(2)} = -\frac{D-1}{2}p_{\perp m}^{(2)}. \tag{38}$$

In particular, for conformally coupled fields all coefficients $q_m^{(1)}$, $q_m^{(2)}$ can be expressed via $\varepsilon_m^{(1)}$, $\varepsilon_m^{(2)}$ using the trace anomaly.

From Eqs. (6), (23), and (27) for the boundary induced parts of the energy-momentum tensor components one obtains

$$q_b(a,r) = -\frac{1}{2\pi a^3 r^n S_D \sigma^{D-1}}\sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} dz\, z^3 \frac{\bar{K}_{\nu_l}(z)}{\bar{I}_{\nu_l}(z)} \frac{F_{\nu_l}^{(q)}\left[I_{\nu_l}(zr/a)\right]}{\sqrt{z^2 - m^2a^2}}, \quad r < a, \tag{39}$$

where for a given function $f(y)$ we have introduced the notations

$$F_{\nu_l}^{(\varepsilon)}\left[f(y)\right] = (1-4\xi)\left[f'^2(y) - \frac{n}{y}f(y)f'(y) + \left(\frac{\nu_l^2}{y^2} - \frac{1+4\xi-2(mr/y)^2}{1-4\xi}\right)f^2(y)\right] \tag{40}$$

$$F_{\nu_l}^{(p)}\left[f(y)\right] = f'^2(y) + \frac{\tilde{\xi}}{y}f(y)f'(y) - \left(1 + \frac{\nu_l^2 + \tilde{\xi}n/2}{y^2}\right)f^2(y),\ \tilde{\xi} = 4(n+1)\xi - n \tag{41}$$

$$F_{\nu_l}^{(p_\perp)}\left[f(y)\right] = (4\xi-1)f'^2(y) - \frac{\tilde{\xi}}{y}f(y)f'(y) + \left[4\xi - 1 + \frac{\nu_l^2(1+\tilde{\xi}) + \tilde{\xi}n/2}{(n+1)y^2}\right]f^2(y). \tag{42}$$

It can be seen that components (39) satisfy Eq. (36) and are finite for $0 < r < a$.

In the case $D = 1$ the angular part in line element (1) is absent and we obtain VEVs for the one – dimensional segment $-a \le x \le a$. In this case $\langle 0_{\rm m}|T_i^k|0_{\rm m}\rangle_{\rm ren} = 0$ and the boundary part remains only. Due to the gamma function in the denominator of the expression for D_l in Eq. (28), now the only nonzero coefficients are $D_0 = D_1 = 1$. Using the standard expressions for the functions $I_{\pm 1/2}(z)$ and $K_{\pm 1/2}(z)$ we obtain the formulae for the vacuum energy-momentum tensor given in Refs. [40, 62]. Note that in this case the vacuum stresses are uniform and do not depend on the parameter ξ. For $D = 2$ and $r \neq 0$ the line element (1) corresponds to the flat background spacetime and coincides with the $2D$ cosmic string geometry. Note that in this case $n = 0$ and from Eq. (10) one has $\nu_l = l/\sigma$.

At the sphere center the main contribution to the boundary induced VEV (39) comes from the summands with $l = 0$ and one has

$$q_b(a,r) \sim -\frac{(2\nu_0 - n)r^{2\nu_0-n-2} f_0^{(q)}}{(2a)^{2\nu_0+1}\pi S_D \sigma^{D-1}\Gamma^2(\nu_0+1)} \int_{ma}^{\infty} dz \frac{z^{2\nu_0+1}}{\sqrt{z^2-m^2a^2}} \frac{\bar{K}_{\nu_0}(z)}{\bar{I}_{\nu_0}(z)}, \quad r \to 0, \tag{43}$$

where the following notations are introduced

$$f_0^{(\varepsilon)} = (1-4\xi)\nu_0, \quad f_0^{(p)} = \frac{\tilde{\xi}}{2}, \quad f_0^{(p_\perp)} = \frac{\nu_0 - 1/2}{D-1}\tilde{\xi}. \tag{44}$$

It follows from Eq. (43) that the boundary induced components are zero at the sphere centre for $n\xi(\sigma^{-2}-1) > 1$, are non-zero constants for $n\xi(\sigma^{-2}-1) = 1$, and are infinite otherwise. These singularities appear because the geometrical characteristics of global monopole spacetime are divergent at the origin. However, note that the corresponding contribution to the total energy of the vacuum inside a sphere coming from ε_b is finite due to the factor r^{D-1} in the volume element. Comparing expressions (37) and (43) we conclude that near the centre the vacuum energy-momentum tensor is dominated by the purely gravitational part.

Expectation values (39) diverge at the sphere surface, $r \to a$. These divergencies are well-known in quantum field theory with boundaries and are investigated for various types of boundary geometries. For the problem under consideration the corresponding asymptotic behavior can be found using the uniform asymptotic expansions for the modified Bessel functions, and the leading terms are determined by the relations

$$p_{b\perp} \sim -\varepsilon_b \sim \frac{Dap_b}{(D-1)(a-r)} \sim \frac{D\Gamma((D+1)/2)(\xi-\xi_D)}{2^D\pi^{(D+1)/2}(a-r)^{D+1}}\left(1-2\delta_{B0}\right). \tag{45}$$

These terms do not depend on mass and parameter σ, and are the same as for a spherical shell in the Minkowski bulk (see, for instance, [40]). For a conformally coupled scalar the coefficients for the leading terms are zero and $\varepsilon, p_\perp \sim (a-r)^{-D}$, $p \sim (a-r)^{1-D}$. In general, asymptotic series can be developed in powers of the distance from the boundary. The corresponding subleading coefficients will depend on the mass, Robin coefficient, and parameter σ. Due to surface divergencies near the sphere surface the total vacuum energy-momentum tensor is dominated by the boundary induced parts q_b.

Now let us consider the VEVs of the energy-momentum tensor in the limit $\sigma \ll 1$ for a fixed $r < a$. For $\xi > 0$ by the calculations similar to those given above for the field square, for the corresponding boundary parts one receives

$$q_b(a,r) \approx \frac{(1-2\delta_{B0})\tilde{\nu}^{3/2}\exp\left[-2\tilde{\nu}\ln(a/r)\right]}{(8\pi)^{1/2}S_D\sigma^{D-1}r^D\sqrt{a^2-r^2}} f_1^{(q)}, \tag{46}$$

where $\tilde{\nu}$ is defined in Eq. (32), and

$$f_1^{(\varepsilon)} = 1-4\xi, \quad f_1^{(p)} = \frac{\tilde{\xi}}{2\tilde{\nu}}, \quad f_1^{(p_\perp)} = \frac{\tilde{\xi}}{D-1}. \tag{47}$$

Note that in this case the boundary induced vacuum stresses are strongly anisotropic: $p_b/p_{\perp b} \sim \sigma \ll 1$. For a minimally coupled scalar , $\xi = 0$, the leading term of the asymptotic expansion over σ comes from the $l = 0$ summand in Eq. (39) with $\nu_l = n/2$. This term behaves as σ^{1-D}. In Fig. 2 (left panel) we have presented the quantities ε_b, p_b, $p_{\perp b}$ inside a spherical shell as functions on r/a in the case of a conformally coupled massless $D = 3$ Dirichlet scalar on background of the global monopole with $\sigma = 0.5$. Note that for these values of parameters $n\xi(\sigma^{-2} - 1) < 1$, and the vacuum energy-momentum tensor components are infinite at the sphere centre.

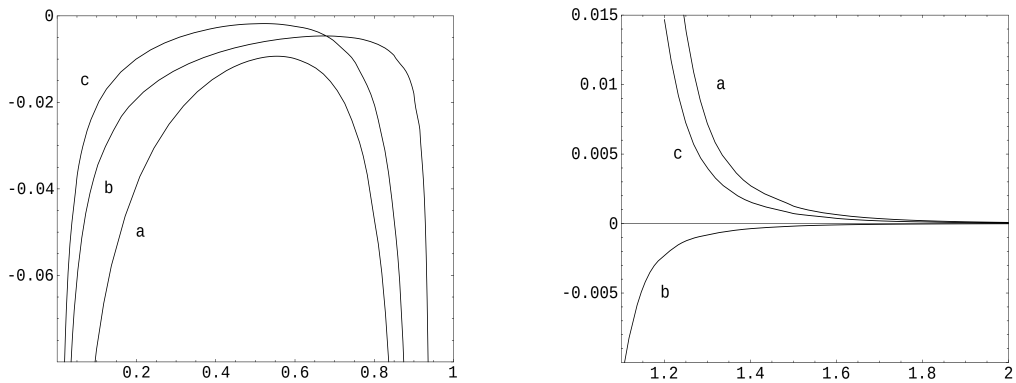

Figure 2: The sphere induced VEVs of the energy density, ε_b (curves a), radial pressure, p_b (curves b) and azimuthal pressure, $p_{\perp b}$ (curves c) multiplied by a^{D+1}, as functions on r/a in the case of a conformally coupled massless $D = 3$ Dirichlet scalar for the regions inside (left panel) and outside (right panel) the sphere. The curves are plotted for $\sigma = 0.5$.

3 Vacuum densities outside a sphere

To obtain the VEV for the energy-momentum tensor outside a sphere we consider first the scalar vacuum in the layer between two concentric spheres with radii a and b, $a < b$. The boundary conditions on these surfaces we will take in the form

$$\left(A_1 + B_1 \frac{\partial}{\partial r} \right) \varphi(x) = 0, \quad r = a, b. \tag{48}$$

The corresponding eigenfunctions can be obtained from Eq. (8) with the replacement

$$J_{\nu_l}(\lambda r) \to g_{\nu_l}(\lambda a, \lambda r) \equiv J_{\nu_l}(\lambda r)\bar{Y}_{\nu_l}(\lambda a) - \bar{J}_{\nu_l}(\lambda a) Y_{\nu_l}(\lambda r), \tag{49}$$

where $Y_{\nu_l}(z)$ is the Neumann function, and the functions with overbars are defined in accordance with Eq. (20). Note that now in the definition of the coefficients in formula (16) we have to take $n^1 = 1$. The functions chosen in this way satisfy the boundary condition on the sphere $r = a$. From the boundary condition on $r = b$ one obtains that the corresponding eigenmodes are solutions to the equation

$$C^{ab}_{\nu_l}(\eta, \lambda a) \equiv \bar{J}_{\nu_l}(\lambda a)\bar{Y}^{(b)}_{\nu_l}(\lambda b) - \bar{J}^{(b)}_{\nu_l}(\lambda b)\bar{Y}_{\nu_l}(\lambda a) = 0, \quad \eta = b/a, \tag{50}$$

with the notation

$$\bar{F}^{(b)}(z) = A_b F(z) + B_b z F'(z), \quad A_b = A_1 - \frac{B_1 n}{2b}, \quad B_b = \frac{B_1}{b}, \tag{51}$$

for a given function $F(z)$. In Appendix A we show that for large values η all roots to Eq. (50) are real. As we are interested in the VEVs for the region outside a single sphere, obtained in the limit $b \to \infty$, this is the case for our consideration below. Let $\gamma_{\nu_l,k} = \lambda a$, $k = 1, 2, \ldots$ be the corresponding positive roots, arranged in ascending order. The coefficients β_α are determined from the normalization condition (14), where now the integration goes over the region between the spheres, $a \le r \le b$. Using the standard formula for integrals involving the product of any two cylinder functions one obtains

$$\beta_\alpha^2 = \frac{\pi^2 \lambda}{4N(m_k)\omega a \sigma^{D-1}} T_{\nu_l}^{ab}(\eta, \lambda a), \tag{52}$$

where $N(m_k)$ comes from the normalization integral (13) and we use the notation

$$T_{\nu_l}^{ab}(\eta, z) = z \left\{ \frac{\bar{J}_{\nu_l}^2(z)}{\bar{J}_{\nu_l}^2(\eta z)} \left[A_b^2 + B_b^2(\eta^2 z^2 - \nu_l^2) \right] - A^2 - B^2(z^2 - \nu_l^2) \right\}^{-1}, \tag{53}$$

with constants A, B, A_b, B_b defined in (16) and (51).

Substituting the eigenfunctions into the mode sum (7) and using the addition formula for the spherical harmonics, the expectation value of the field product is presented in the form

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \frac{\pi^2 (rr')^{-n/2}}{4naS_D\sigma^{D-1}} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos\theta) \sum_{k=1}^{\infty} h(\gamma_{\nu_l,k}) T_{\nu_l}^{ab}(\eta, \gamma_{\nu_l,k}), \tag{54}$$

with the function

$$h(z) = \frac{z e^{i\sqrt{z^2/a^2+m^2}(t'-t)}}{\sqrt{z^2+m^2a^2}} g_{\nu_l}(z, zr/a) g_{\nu_l}(z, zr'/a). \tag{55}$$

To sum over k we will use the summation formula [40, 52, 53]

$$\begin{aligned} \frac{\pi^2}{2} \sum_{k=1}^{\infty} h(\gamma_{\nu,k}) T_\nu^{ab}(\eta, \gamma_{\nu,k}) \quad &= \quad \int_0^\infty \frac{h(x)dx}{\bar{J}_\nu^2(x) + \bar{Y}_\nu^2(x)} - r_\nu[h(z)] \\ &\quad -\frac{\pi}{4} \int_0^\infty \frac{\bar{K}_\nu^{(b)}(\eta x)}{\bar{K}_\nu(x)} \frac{\left[h(xe^{\pi i/2}) + h(xe^{-\pi i/2}) \right] dx}{\bar{K}_\nu(x)\bar{I}_\nu^{(b)}(\eta x) - \bar{K}_\nu^{(b)}(\eta x)\bar{I}_\nu(x)}, \end{aligned} \tag{56}$$

where we have assumed that all zeros for the function (50) are real. In this formula the term

$$r_\nu[h(z)] = \pi \sum_{k,s=1,2} \operatorname{Res}_{z=z_{\nu,k}^{(s)}} \left[\frac{\bar{H}_\nu^{(sb)}(\eta z) h(z)}{\bar{H}_\nu^{(s)}(z) C_\nu^{ab}(\eta, z)} \right] = \pi i \sum_{k,s=1,2} \frac{(-1)^{s+1} h(z_{\nu,k}^{(s)})}{\bar{H}_\nu^{(s)\prime}(z_{\nu,k}^{(s)}) \bar{J}_\nu(z_{\nu,k}^{(s)})} \tag{57}$$

comes from the residues at the possible zeros $z = z_{\nu,k}^{(s)}$ of the function $\bar{H}_\nu^{(s)}(z)$, $s = 1, 2$, with $0 < \arg z_{\nu,k}^{(1)} < \pi/2$, $z_{\nu,k}^{(2)} = z_{\nu,k}^{(1)*}$. Here and below $H_\nu^{(s)}(z)$, $s = 1, 2$ are the Hankel functions. Note that, as we have shown in Appendix A, under the condition (17) the function $\bar{K}_\nu(z)$ has no real zeros and, hence, the function $\bar{H}_\nu^{(s)}(z)$ has no purely imaginary zeros. In the case of function (55) the corresponding conditions for (56) are satisfied if $r + r' + |t - t'| < 2b$. Note that this is the case in the coincidence limit for the region under consideration. Applying to the sum over k in Eq. (54) formula (56), for the corresponding Wightman function one obtains

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \frac{1}{2naS_D\sigma^{D-1}} \sum_{l=0}^{\infty} \frac{2l+n}{(rr')^{n/2}} C_l^{n/2}(\cos\theta) \left\{ \int_0^\infty \frac{h(z)dz}{\bar{J}_{\nu_l}^2(z) + \bar{Y}_{\nu_l}^2(z)} - r_{\nu_l}[h(z)] \right. \tag{58}$$

$$-\frac{2}{\pi}\int_{ma}^{\infty}\frac{zdz}{\sqrt{z^2-a^2m^2}}\frac{\bar{K}_{\nu_l}^{(b)}(\eta z)}{\bar{K}_{\nu_l}(z)}\frac{G_{\nu_l}(z,zr/a)G_{\nu_l}(z,zr'/a)}{\bar{K}_{\nu_l}(z)\bar{I}_{\nu_l}^{(b)}(\eta z)-\bar{K}_{\nu_l}^{(b)}(\eta z)\bar{I}_{\nu_l}(z)}\cosh\left[\sqrt{z^2/a^2-m^2}(t'-t)\right]\Bigg\},$$

where we have introduced notation

$$G_\nu(z,y)=I_\nu(y)\bar{K}_\nu(z)-\bar{I}_\nu(z)K_\nu(y), \tag{59}$$

with the modified Bessel functions.

To obtain the Wightman function outside a single sphere on background of the global monopole geometry, let us consider the limit $b\to\infty$. In this limit the second integral on the right of Eq. (58) tends to zero (for large values of the ratio b/a the subintegrand is proportional to $e^{-2bz/a}$), whereas the first integral and the term $r_{\nu_l}[h(z)]$ do not depend on b. It follows from here that the quantity

$$\begin{aligned}\langle 0|\varphi(x)\varphi(x')|0\rangle &= \frac{1}{2naS_D\sigma^{D-1}}\sum_{l=0}^{\infty}\frac{2l+n}{(rr')^{n/2}}C_l^{n/2}(\cos\theta) \\ &\times \left\{\int_0^\infty\frac{zdz}{\sqrt{z^2+m^2a^2}}\frac{g_{\nu_l}(z,zr/a)g_{\nu_l}(z,zr'/a)}{\bar{J}_{\nu_l}^2(z)+\bar{Y}_{\nu_l}^2(z)}e^{i\sqrt{z^2/a^2+m^2}(t'-t)}-r_{\nu_l}[h(z)]\right\}\end{aligned} \tag{60}$$

is the Wightman function for the exterior region of a single sphere with radius a. To find the boundary induced VEVs we have to subtract the corresponding part for the unbounded monopole geometry which, as we saw in previous section, can be presented in the form (22). Using the relation

$$\frac{g_\nu(z,zr/a)g_\nu(z,zr'/a)}{\bar{J}_\nu^2(z)+\bar{Y}_\nu^2(z)}-J_\nu(zr/a)J_\nu(zr'/a)=-\frac{1}{2}\sum_{s=1}^{2}\frac{\bar{J}_\nu(z)}{\bar{H}_\nu^{(s)}(z)}H_\nu^{(s)}(zr/a)H_\nu^{(s)}(zr'/a) \tag{61}$$

one obtains

$$\begin{aligned}\langle 0|\varphi(x)\varphi(x')|0\rangle-\langle 0_{\rm m}|\varphi(x)\varphi(x')|0_{\rm m}\rangle &= -\frac{1}{4nS_D\sigma^{D-1}}\sum_{l=0}^{\infty}\frac{2l+n}{(rr')^{n/2}}C_l^{n/2}(\cos\theta) \\ \times\left\{\sum_{s=1}^{2}\int_0^\infty dz\,z\frac{e^{i\sqrt{z^2+m^2}(t'-t)}}{\sqrt{z^2+m^2}}\frac{\bar{J}_{\nu_l}(za)}{\bar{H}_{\nu_l}^{(s)}(za)}H_{\nu_l}^{(s)}(zr)H_{\nu_l}^{(s)}(zr')+\frac{2}{a}r_{\nu_l}[h(z)]\right\}.\end{aligned} \tag{62}$$

In this formula we can rotate the integration contour on the right by the angle $\pi/2$ for $s=1$ and by the angle $-\pi/2$ for $s=2$. It can be easily seen that the residue terms coming from the poles at the zeros of the functions $\bar{H}_\nu^{(s)}(za)$ cancel the second summand in the figure braces in Eq. (62). Further, the integrals over the segments $(0,ima)$ and $(0,-ima)$ of the imaginary axis cancel out and after introducing the Bessel modified functions one obtains

$$\begin{aligned}\langle\varphi(x)\varphi(x')\rangle_b &= -\frac{1}{\pi naS_D\sigma^{D-1}}\sum_{l=0}^{\infty}\frac{2l+n}{(rr')^{n/2}}C_l^{n/2}(\cos\theta) \\ &\times \int_{ma}^{\infty}dz\,z\frac{\bar{I}_{\nu_l}(z)}{\bar{K}_{\nu_l}(z)}\frac{K_{\nu_l}(zr/a)K_{\nu_l}(zr'/a)}{\sqrt{z^2-m^2a^2}}\cosh\left[\sqrt{z^2/a^2-m^2}(t'-t)\right].\end{aligned} \tag{63}$$

This formula differs from the one for the interior region by the replacements $I_{\nu_l}\to K_{\nu_l}$, $K_{\nu_l}\to I_{\nu_l}$. For the boundary induced VEV of the field square in the outside region this leads to the formula

$$\langle\varphi^2(x)\rangle_b=-\frac{1}{\pi ar^nS_D\sigma^{D-1}}\sum_{l=0}^{\infty}D_l\int_{ma}^{\infty}dz\,z\frac{\bar{I}_{\nu_l}(z)}{\bar{K}_{\nu_l}(z)}\frac{K_{\nu_l}^2(zr/a)}{\sqrt{z^2-m^2a^2}}, \tag{64}$$

where D_l is defined in accord with Eq. (28). This expression is finite for all values $r > a$ and diverges at the sphere surface. The leading term in the corresponding asymptotic expansion over $r - a$ has the form (45) with replacement $a - r \rightarrow r - a$. In the case of the Dirichlet boundary condition $\langle \varphi^2(x) \rangle_b$ is monotonic increasing negative function on r.

For the case of a massless scalar the asymptotic behavior of boundary part (64) at large distances from the sphere can be obtained by introducing a new integration variable $y = zr/a$ and expanding the subintegrand in terms of a/r. The leading contribution for the summand with a given l has an order $(a/r)^{2\nu_l + D - 1}$ (assuming that $A \neq \nu_l B$) and the main contribution comes from the $l = 0$ term. Using the value for the standard integral involving the product of the functions K_ν [61], the leading term for the asymptotic expansion over a/r can be presented in the form

$$\langle \varphi^2(x) \rangle_b \approx -\frac{\nu_0 \Gamma(2\nu_0 + 1/2)\Gamma(\nu_0 + 1/2)}{2^{2\nu_0+1}(a\sigma)^{D-1} S_D \Gamma^3(\nu_0 + 1)} \frac{A + B\nu_0}{A - B\nu_0} \left(\frac{a}{r}\right)^{2\nu_0 + D - 1}. \tag{65}$$

Now comparing this with Eq. (37), we see that for $\nu_0 > 0$ the VEV of the field square at large distances from the sphere is dominated by the purely gravitational part. In the limit $\sigma \ll 1$ the leading terms in the asymptotic expansion of Eq. (64) are derived by the way similar to that used above for the inside region. For $\xi > 0$ the corresponding formula is obtained from Eq. (32) with replacements $a^2 - r^2 \rightarrow r^2 - a^2$ under the square root and $a/r \rightarrow r/a$ under the ln function. In the case $\xi = 0$ in Eq. (33) we have to make replacements $I_{\nu_l} \rightarrow K_{\nu_l}$, $K_{\nu_l} \rightarrow I_{\nu_l}$. In the right panel of Fig. 1 we have presented the dependence of the boundary induced VEV (64) on r/a for the region outside the sphere in the case of a conformally coupled ($\xi = \xi_D$) massless Dirichlet scalar in $D = 3$ spatial dimensions. The curves correspond to the values $\sigma = 1$ (a), $\sigma = 0.5$ (b), $\sigma = 0.2$ (c).

As in the interior case, the vacuum energy-momentum tensor is diagonal and the corresponding components can be presented in the form of a sum of purely gravitational and boundary parts, Eq. (35). The parts of these components induced by the presence of a spherical shell are given by formulae

$$q_b(a, r) = -\frac{1}{2\pi a^3 r^n S_D \sigma^{D-1}} \sum_{l=0}^{\infty} D_l \int_{ma}^{\infty} dz\, z^3 \frac{\bar{I}_{\nu_l}(z)}{\bar{K}_{\nu_l}(z)} \frac{F_{\nu_l}^{(q)}[K_{\nu_l}(zr/a)]}{\sqrt{z^2 - m^2 a^2}}, \quad q = \varepsilon,\, p,\, p_\perp, \quad r > a, \tag{66}$$

where the functions $F_{\nu_l}^{(q)}[f(y)]$ are given by relations (40), (41) and (42). As for the interior components, the quantities (66) diverge at the sphere surface, $r = a$. The leading terms of the asymptotic expansions are determined by same formulae (45) with the replacement $a - r \rightarrow r - a$. When $D = 1$ from Eq. (66) we obtain the expectation values for the one dimensional semi-infinite region $r > a$, given in Refs. [40, 62].

For large distances from the sphere, $r \gg a$, the main contribution into the VEV of the energy-momentum tensor comes from the $l = 0$ summand. The leading terms of the asymptotic expansions have the form

$$q_b \approx -\frac{2^{-2\nu_0} a^{-D-1} \sigma^{1-D}}{\pi \nu_0 S_D \Gamma^2(\nu_0)} \frac{A + B\nu_0}{A - B\nu_0} \left(\frac{a}{r}\right)^{2\nu_0 + D + 1} \int_0^{\infty} dz z^{2\nu_0 + 2} F_{\nu_0}^{(q)}[K_{\nu_0}(z)], \tag{67}$$

where ν_0 is determined by formula (17). Note that the integrals in this formula can be evaluated using the value for the integrals involving the product of the functions K_ν given in Ref. [61]. As we see, for $\nu_0 > 0$ and for large distances from the sphere the vacuum energy-momentum tensor is dominated by the purely gravitational part. In Fig. 2 (right panel) we have presented the sphere induced quantities ε_b, p_b, $p_{\perp b}$ for the region outside a spherical shell as functions on r/a in the case of a conformally coupled massless $D = 3$ Dirichlet scalar field on background of the global monopole with the solid angle deficit $\sigma = 0.5$.

4 Conclusion

In the present paper we have investigated the Casimir densities induced by a spherical shell on background of the $D+1$-dimensional global monopole spacetime described by the metric (1). The case of a massive scalar field with general curvature coupling parameter and satisfying the Robin boundary condition on the sphere is considered. All calculations are made at zero temperature and we assume that the boundary conditions are frequency independent. The latter means no dispersive effects are taken into account. To obtain the expectation values for the energy-momentum tensor we first construct the positive frequency Wightmann function (note that the Wightmann function is also important in considerations of the response of a particle detector at a given state of motion [33]). The application of the generalized Abel-Plana formula to the mode sum over zeros of the corresponding combinations of the cylinder functions allows us to extract the part due to the global monopole geometry without boundaries and to present the sphere induced part in terms of integrals which are exponentially convergent in the coincidence limit at any strictly interior or exterior points. The polarization of the scalar vacuum by the global monopole without boundaries is investigated in a number of previous papers [45]–[49], and our main interest in this paper are the sphere induced parts for the VEVs (Casimir densities). The expectation values for the energy-momentum tensor are obtained by applying on the corresponding Wightman function a certain second-order differential operator and taking the coincidence limit. These quantities diverge as the boundary is approached. Surface divergences are well known in quantum field theory with boundaries and are investigated near an arbitrary shaped smooth boundary. They lead to divergent global quantities, such as the total energy or vacuum forces acting on the sphere and additional renormalization procedure is needed. The total Casimir energy for a spherical shell on the global monopole background is investigated in Refs. [50, 47] using the zeta function regularization method.

The boundary induced expectation values of the field square and energy-momentum tensor for the region inside a spherical shell are given by formulae (27) and (39) respectively. These expressions are finite at interior points and diverge on the sphere surface. The leading term in the corresponding asymptotic expansions is the same as for a sphere in the Minkowski bulk and is zero for a conformally coupled scalar. The coefficients for the subleading asymptotic terms will depend on the boundary curvature, Robin coefficient, mass and the parameter σ describing the solid angle deficit. In section 3 to deal with discrete modes, first we consider the scalar vacuum in the spherical layer between two surfaces. The quantities characterizing the vacuum outside a single sphere are obtained from this case in the limit when the radius of the outer sphere tends to infinity. Subtracting the parts corresponding to the space without boundaries, for the boundary induced parts of the Wightman function and vacuum densities we derive formulae (63), (64) and (66). They differ from the ones for the interior region by the replacements $I_{\nu_l} \to K_{\nu_l}$, $K_{\nu_l} \to I_{\nu_l}$. Near the sphere centre and at large distances from the sphere the VEVs for the field square and energy-momentum tensor are dominated by the purely gravitational parts corresponding to the monopole geometry without boundaries. For the points near the sphere surface the boundary induced parts in the VEVs dominate. We have also investigated the VEVs in the limit of small values for the parameter σ, describing the solid angle deficit, $\sigma \ll 1$. In this limit, corresponding to strong gravitational fields, for $\xi > 0$ the boundary induced parts behave as $\langle \varphi^2 \rangle_b \sim \sigma^{3/2-D} \exp(-\gamma/\sigma)$ and $\varepsilon_b \sim p_{\perp b} \sim p_b/\sigma \sim \sigma^{-D-1/2} \exp(-\gamma/\sigma)$, with $\gamma = 2\sqrt{n(n+1)\xi}|\ln(a/r)|$, and the boundary induced stresses are strongly anisotropic. For a minimally coupled scalar ($\xi = 0$) and $\sigma \ll 1$ we have shown that $\langle \varphi^2 \rangle_b \sim q_b \sim \sigma^{1-D}$, $q = \varepsilon, p_\perp, p$. As an illustration of general results, in Fig. 1 and Fig. 2 we have plotted the sphere induced VEVs of the field square and energy-momentum tensor components as functions on r/a for a conformally coupled Dirichlet scalar in $D = 3$ spatial dimensions. Note that in this case the leading terms in the asymptotic expansions of the vacuum energy-momentum tensor components near the boundary vanish. The VEV of the field square is negative everywhere and the energy density is negative/positive inside/outside the sphere.

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A On the zeros of the function $C_\nu^{ab}(\eta, z)$

Here we will first show that for real values of the coefficients A, B, A_b, B_b in (16) and (51) all roots of the equation

$$C_\nu^{ab}(\eta, z) \equiv \bar{J}_\nu(z)\bar{Y}_\nu^{(b)}(\eta z) - \bar{Y}_\nu(z)\bar{J}_\nu^{(b)}(\eta z) = 0, \quad \eta > 1 \tag{68}$$

are real or purely imaginary. To see this note that from the relation $(C_\nu^{ab}(\eta, z))^* = C_\nu^{ab}(\eta, z^*)$ it follows that if z is a root to (68) when z^* is also a root. Further, from the Bessel equation for the function $g_\nu(z, zx)$ defined in (49) (considered as a function on x) the following orthogonality relation can be derived

$$\int_1^\eta dx\, x g_\nu(z_1, z_1 x) g_\nu(z_2, z_2 x) = 0, \quad z_1^2 \neq z_2^2, \tag{69}$$

for two roots z_1 and z_2 of Eq. (68). Assuming $z_1^{*2} \neq z_1^2$ and taking in this relation $z_2 = z_1^*$, we obtain a contradiction:

$$\int_1^\eta dx\, x |g_\nu(z_1, z_1 x)|^2 = 0, \tag{70}$$

and, hence, $z_1^{*2} = z_1^2$. This means that the zeros of the function $C_\nu^{ab}(\eta, z)$ are real or purely imaginary.

In this paper we are interested in the limit $\eta \to \infty$. Let us show that for large values η all zeros are real. For the purely imaginary z, $z = ye^{\pi i/2}$ with real y, the function $C_\nu^{ab}(\eta, z)$ can be expressed via the Bessel modified functions:

$$C_\nu^{ab}(\eta, ye^{\pi i/2}) = \frac{2}{\pi}\left[\bar{K}_\nu(y)\bar{I}_\nu^{(b)}(\eta y) - \bar{I}_\nu(y)\bar{K}_\nu^{(b)}(\eta y)\right]. \tag{71}$$

For large η the contribution of the second term in the square brackets is exponentially suppressed and one has

$$C_\nu^{ab}(\eta, ye^{\pi i/2}) \approx \frac{2}{\pi}\bar{K}_\nu(y)\bar{I}_\nu^{(b)}(\eta y), \quad \eta \gg 1. \tag{72}$$

From the asymptotic expansion of the function $I_\nu(z)$ for large values of the argument it can be seen that for large η the function $\bar{I}_\nu^{(b)}(\eta y)$ has no zeros. Further, using the recurrent relations for the Bessel modified functions and expressions (16) for the coefficients A and B with $n^1 = 1$, we can see that

$$\bar{K}_\nu(y) = -\frac{B_1}{a}\left[\left(\nu + \frac{n}{2} - \frac{A_1 a}{B_1}\right)K_\nu(y) + yK_{\nu-1}(y)\right]. \tag{73}$$

Under the condition (17) with $\nu \geq \nu_0$ the expression in the square brackets of this equation is always positive and the function $\bar{K}_\nu(y)$ has no real zeros. As a result we conclude that for large η the function $C_\nu^{ab}(\eta, ye^{\pi i/2})$ has no real zeros with respect to y. Therefore, for sufficiently large η all zeros of the function $C_\nu^{ab}(\eta, z)$ are real.

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